Univerzitet u Zenici
Filozofski fakultet
Odsjek: Matematika i informatika
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Pismeni ispit iz predmeta Linearna algebra

Uputa: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. U $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ zadani su podprostori

$$\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a - 2b = 0,\ a + c + d = 0 \right\} \quad \text{i}$$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + c = 0,\ a - 2b + d = 0 \right\}.$$

Odrediti po jednu bazu za $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} + \mathcal{N}$ i $\mathcal{M} \cap \mathcal{N}$.

2. Zadan je linearni operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ svojom matricom $T = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ u kanonskoj bazi $\{\vec{i}, \vec{j}\}$. Neka su $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = \vec{i} - 2\vec{j}$.

(*a*) Odrediti $T(\vec{a})$, $T(\vec{b})$.

(*b*) Za koje $\alpha \in \mathbb{R}$ su vektori $T(\vec{a})$, $T(\vec{a} + \alpha\vec{b})$ kolinearni?

3. Neka je T linearan operator na prostoru $\mathbb{R}^2$ koji vektor najprije reflektuje (zrcali) s obzirom na pravac $y = -x$, zatim ga rotira za ugao $\frac{\pi}{4}$ oko koordinatnog početka (oko izvorišta) u negativnom smjeru, te zatim reflektuje (zrcali) s obzirom na pravac $y = x$. Naći matricu (matricu koordinata) operatora T u bazi $\mathcal{B} = \left\{ 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

4. U unitarnom prostoru $\mathbb{R}^4$, sa skalarnim proizvodom

$$\langle x, y \rangle = x_1y_1 + 2x_2y_2 + x_3y_3 + 2x_4y_4$$

zadan je podprostor $\mathcal{V}$ razapet (generisan) vektorima $v_1 = (1, 0, 1, 0)^\top$ i $v_2 = (1, 0, 1, 1)^\top$. Prikažite vektor $x = (4, 2, 2, 4)^\top$ u obliku $x = v + w$, gdje je $v \in \mathcal{V}$, $w \in \mathcal{V}^\perp$.

(#) U $Mat_{2\times 2}(\mathbb{R})$ zadani su podprostori

$$\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a-2b=0,\ a+c+d=0 \right\} \quad \text{i}$$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+c=0,\ a-2b+d=0 \right\}.$$

Odrediti po jednu bazu za $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M}+\mathcal{N}$ i $\mathcal{M}\cap\mathcal{N}$.

Rj.

$\boxed{Mat_{2\times 2} \cong \mathbb{R}^4 \text{ kao vektorski prostori}}$

Primjetimo da matricu $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ možemo tumačiti i kao vektor $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, pa da bi odredili baze za $\mathcal{M}$ i $\mathcal{N}$, puno je jednostavnije posmatrati sljedeća dva vektorska podprostora prostora $\mathbb{R}^4$.

$$\mathcal{M}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, a-2b=0,\ a+c+d=0 \right\} \quad \text{i}$$

$$\mathcal{N}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, a+c=0,\ a-2b+d=0 \right\}.$$

Svedimo sad prostore $\mathcal{M}'$ i $\mathcal{N}'$ na jezgra nekih matrica

$$\mathcal{M}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \underbrace{\begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \ker(A)$$

$$\mathcal{N}' = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}}_{=B} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \ker(B)$$

Znamo da kolone iz općeg rješenja sistema $Ax=0$ formiraju bazu za $\ker(A)$.

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right) \overset{II_v - I_v}{\sim} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right) \overset{I_v + II_v}{\sim} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right) \end{array}$$

$\Rightarrow$ dvije promjenjive uzimamo proizvoljno npr. $c=s$, $d=t$

Rješenja su oblika

$$a+c+d=0$$
$$2b+c+d=0$$
$$a=-c-d$$
$$2b=-c-d$$
$$b=-\frac{1}{2}(c+d)$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s-t \\ -\frac{1}{2}s-\frac{1}{2}t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ -1/2 \\ 0 \\ 1 \end{pmatrix} t$$

Baza za M je

$$\left\{ \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \right\}$$

Slično, posmatramo $\ker(B)$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \end{array}\right) \overset{II_v - I_v}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \end{array}\right) \Rightarrow$$

dvije promjenjive uzimamo proizvoljno npr. $c=s$, $d=t$

$$a+c=0$$
$$-2b-c+d=0$$
$$a=-c$$
$$-2b=c-d$$
$$a=-c$$
$$b=-\frac{1}{2}c+\frac{1}{2}d$$

Rješenja su oblika

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s \\ -\frac{1}{2}s+\frac{1}{2}t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} t$$

Baza za N je

$$\left\{ \begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \right\}.$$

Prema definiciji

$$\mathcal{M}+\mathcal{N}=\{m+n \mid m\in\mathcal{M} \text{ i } n\in\mathcal{N}\}$$

Pa da bi našli bazu za $\mathcal{M}+\mathcal{N}$ prvo nađimo bazu za $\mathcal{M}'+\mathcal{N}'$ tj. pronađimo linearno nezavisan skup iz unije baza za $\mathcal{M}'$ i $\mathcal{N}'$. Ili iz definicije od $\mathcal{M}+\mathcal{N}$.

Znamo da

Ako $\mathcal{S}_{\mathcal{X}}, \mathcal{S}_{\mathcal{Y}}$ generišu $\mathcal{X}, \mathcal{Y}$ tada $\mathcal{S}_{\mathcal{X}}\cup\mathcal{S}_{\mathcal{Y}}$ generiše $\mathcal{X}+\mathcal{Y}$.

Primjetimo da

$$\mathcal{M}'=\ker(A)=\operatorname{span}\left\{\begin{pmatrix}-1\\-\frac{1}{2}\\1\\0\end{pmatrix},\begin{pmatrix}-1\\-\frac{1}{2}\\0\\1\end{pmatrix}\right\}=\operatorname{im}\left(\begin{bmatrix}-1&-1\\-\frac{1}{2}&-\frac{1}{2}\\1&0\\0&1\end{bmatrix}\right)$$

$$\mathcal{N}'=\ker(B)=\operatorname{span}\left\{\begin{pmatrix}-1\\-\frac{1}{2}\\1\\0\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\1\end{pmatrix}\right\}=\operatorname{im}\left(\begin{bmatrix}-1&0\\-\frac{1}{2}&\frac{1}{2}\\1&0\\0&1\end{bmatrix}\right)$$

Znamo $\operatorname{im}(A^T)=\operatorname{im}(B^T)$ akko $A \overset{\text{red}}{\sim} B$

$$\begin{pmatrix}-1&-\frac{1}{2}&1&0\\-1&-\frac{1}{2}&0&1\\-1&-\frac{1}{2}&1&0\\0&\frac{1}{2}&0&1\end{pmatrix}\underset{III_v-I_v}{\overset{II_v-I_v}{\sim}}\begin{pmatrix}-1&-\frac{1}{2}&1&0\\0&0&-1&1\\0&0&0&0\\0&\frac{1}{2}&0&1\end{pmatrix}\overset{II_v\leftrightarrow IV_v}{\sim}\begin{pmatrix}-1&-\frac{1}{2}&1&0\\0&\frac{1}{2}&0&1\\0&0&0&0\\0&0&-1&1\end{pmatrix}$$

Baza za $\mathcal{M}+\mathcal{N}$ je

$$\left\{\begin{pmatrix}-1&-\frac{1}{2}\\1&0\end{pmatrix},\begin{pmatrix}0&\frac{1}{2}\\0&1\end{pmatrix},\begin{pmatrix}0&0\\-1&1\end{pmatrix}\right\}$$

$$M' \cap N' = \{ x \in \mathbb{R}^4 \mid x \in M' \text{ i } x \in N' \}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; a-2b=0,\ a+c+d=0,\ a+c=0\ \ a-2b+d=0 \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \underbrace{\begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}}_{=C} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \ker(C)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \end{array}\right) \underset{IV_v - I_v}{\overset{II_v - I_v,\ III_v - I_v}{\sim}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{III_v - II_v}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \underset{II_v + III_v}{\overset{IV_v + III_v}{\sim}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

1 promenjivu uzimamo proizvoljno

$$\begin{aligned} a - 2b &= 0 \\ 2b + c &= 0 \\ -d &= 0 \end{aligned}$$

$c = s$
$s \in \mathbb{R}$

$a = 2b$

$2b = -c \Rightarrow b = -\frac{1}{2}s$

$a = -s$

rješenje je

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s \\ -\frac{1}{2}s \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} s$$

Baza za $M \cap N$ je $\left\{ \begin{pmatrix} -1 & -1/2 \\ 1 & 0 \end{pmatrix} \right\}$.

(#) Zadan je linearni operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ svojom matricom $T = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ u kanonskoj bazi $\{\vec{i}, \vec{j}\}$.

Neka su $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = \vec{i} - 2\vec{j}$.

(a) Odrediti $T(\vec{a})$, $T(\vec{b})$.

(b) Za koje $\alpha \in \mathbb{R}$ su vektori $T(\vec{a})$, $T(\vec{a} + \alpha\vec{b})$ kolinearni?

Rj.

Kanonsku bazu $\{\vec{i}, \vec{j}\}$ u $\mathbb{R}^2$ možemo pisati i kao $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Šta znači da je linearni operator T $T: \mathbb{R}^2 \to \mathbb{R}^2$ dat svojom matricom $T = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ u kanonskoj bazi $\{\vec{i}, \vec{j}\}$? Prisjetimo se

Matrica koordinata

Neka su $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ i $\mathcal{B}' = \{v_1, v_2, \dots, v_n\}$, redom, baze za $\mathcal{U}$ i $\mathcal{V}$. Matrica koordinata od $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ u odnosu na par $(\mathcal{B}, \mathcal{B}')$ je definisana kao $m \times n$ matrica

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

Ako je samo jedna baza u igri umjesto $[T]_{\mathcal{B}\mathcal{B}}$ koristimo $[T]_{\mathcal{B}}$.

Ako kanonsku bazu $\{\vec{i}, \vec{j}\}$ označimo sa φ tj:

$$\varphi = \Big\{ \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{i}}, \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{j}} \Big\}$$

to imamo

$$T = [T]_{\varphi} = \begin{pmatrix} | & | \\ [T(\vec{i})]_{\varphi} & [T(\vec{j})]_{\varphi} \\ | & | \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$[T(\vec{i})]_{\varphi} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{tj.} \qquad T(\vec{i}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$[T(\vec{j})]_{\varphi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \qquad T(\vec{j}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

a)

$$T(\vec{a}) = T(\vec{i} + \vec{j}) = T(\vec{i}) + T(\vec{j}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$T(\vec{b}) = T(\vec{i} - 2\vec{j}) = T(\vec{i}) - 2T(\vec{j}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

b)

$$T(\vec{a} + \alpha \vec{b}) = T(\vec{a}) + \alpha T(\vec{b}) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1-\alpha \\ 2-\alpha \end{pmatrix}$$

$$T(\vec{a}) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Neki Vektori $\vec{t}_1$ i $\vec{t}_2$ su kolinearni ako su linearno zavisni ili drugim riječima ako postoji realan broj k t.d. $\vec{t}_1 = k\vec{t}_2$.

U našem slučaju npr. tražimo broj η takav da je

$$T(\vec{a} + \alpha \vec{b}) = \eta \, T(\vec{a})$$

tj.

$$\begin{pmatrix} -1-\alpha \\ 2-\alpha \end{pmatrix} = \eta \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$-1-\alpha = -\eta$$
$$2-\alpha = 2\eta$$

$$\alpha - \eta = -1$$
$$\alpha + 2\eta = 2$$

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 2 & 2 \end{array}\right] \overset{II_v - I_v}{\sim} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 3 & 3 \end{array}\right] \overset{II_v : 3}{\sim} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\overset{I_v + II_v}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} \alpha = 0 \\ \eta = 1 \end{array}$$

Za $\alpha = 0$ vektori $T(\vec{a})$ i $T(\vec{a} + \alpha \vec{b})$ su kolinearni.

(#) Neka je T linearan operator na prostoru $\mathbb{R}^2$ koji vektor najprije reflektuje (zrcali) s obzirom na pravac $y=-x$, zatim ga rotira za ugao $\frac{\pi}{4}$ oko koordinatnog početka (oko izvorišta) u negativnom smjeru, te zatim reflektuje (zrcali) s obzirom na pravac $y=x$. Naći matricu (matricu koordinata) operatora T u bazi $\mathcal{B}=\left\{2\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}, -\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

Rj: Prisjetimo se

Matrica koordinata

Neka su $\mathcal{B}=\{u_1,u_2,\dots,u_n\}$ i $\mathcal{B}'=\{v_1,v_2,\dots,v_m\}$, redom, baze za $\mathcal{U}$ i $\mathcal{V}$. Matrice koordinata od $T\in\mathcal{L}(\mathcal{U},\mathcal{V})$ u odnosu na par $(\mathcal{B},\mathcal{B}')$ je definisana kao $m\times n$ matrica

$$[T]_{\mathcal{B}\mathcal{B}'}=\begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

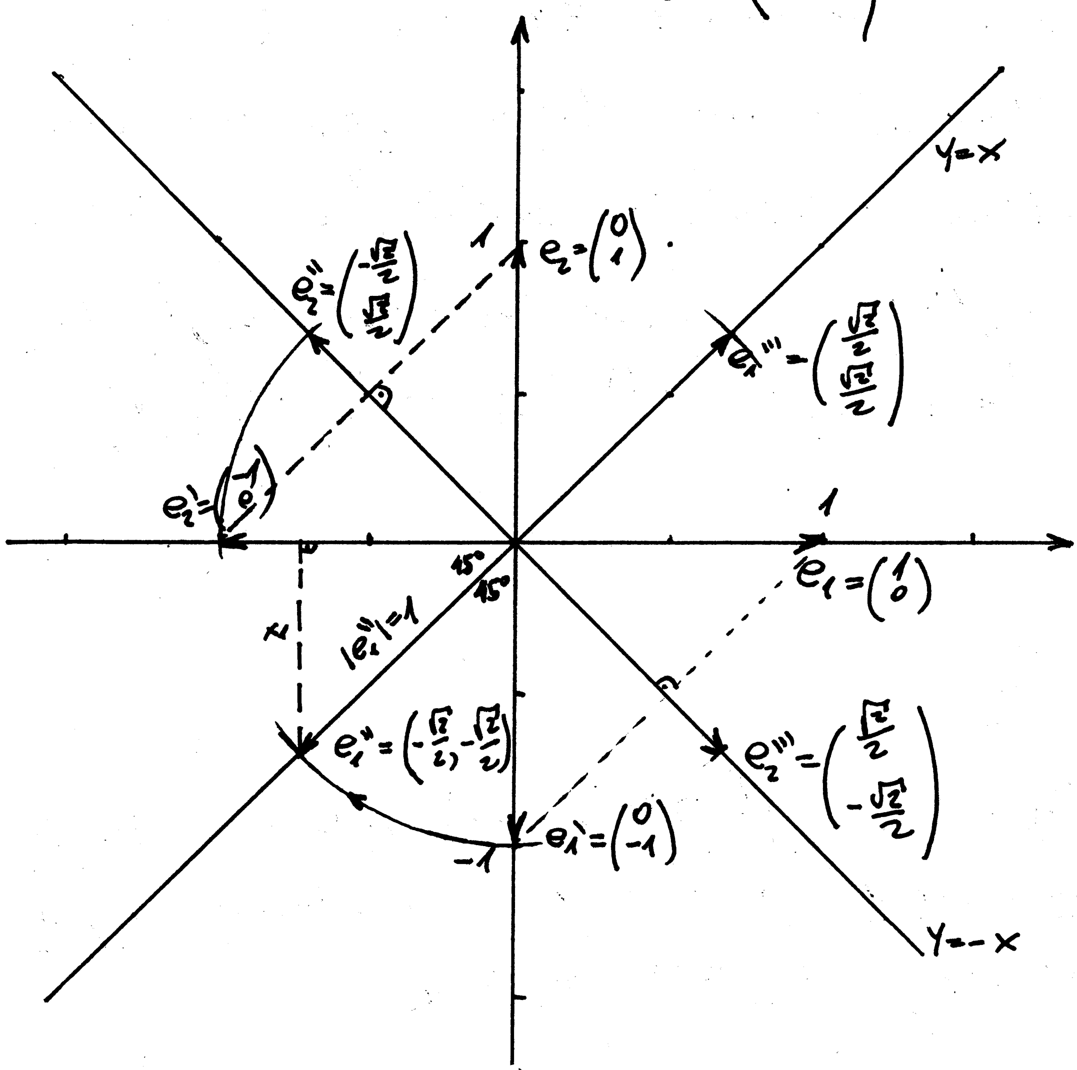
Kada je T linearni operator na $\mathcal{U}$, tada je u igri samo jedna baza, i koristimo $[T]_{\mathcal{B}}$ umjesto $[T]_{\mathcal{B}\mathcal{B}}$.

U našem slučaju $\mathcal{B}=\left\{\begin{pmatrix}2\\-1\end{pmatrix},\begin{pmatrix}-1\\2\end{pmatrix}\right\}$.

Puno je lakše prvo odrediti matricu linearnog operatora u odnosu na standardnu bazu $\mathcal{S}=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

$$\varphi = \{ \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{e_2} \}$$

$$[T]_\varphi = \begin{pmatrix} | & | \\ [T(e_1)]_\varphi & [T(e_2)]_\varphi \\ | & | \end{pmatrix}$$



$\sin 45^\circ = \frac{x_1}{1}$

$x_1 = \frac{\sqrt{2}}{2}$

Sa slike nije teško izračunati da je

$$[T(e_1)]_\varphi = \underbrace{\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}}_{=T(e_1)}, \qquad [T(e_2)]_\varphi = \underbrace{\begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix}}_{=T(e_2)}$$

Prisjetimo se

Djelovanje kao matrično množenje

Neka je $T \in \mathcal{L}(U,V)$ i neka su $\mathcal{B}$ i $\mathcal{B}'$ baze za U i V redom. Za $\forall u \in U$

$$[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'} \cdot [u]_{\mathcal{B}}$$

Nama treba matrica koordinata operatora T u bazi $\mathcal{B}$

$$[T]_{\mathcal{B}} = \begin{pmatrix} | & | \\ \left[T\begin{pmatrix}2\\-1\end{pmatrix}\right]_{\mathcal{B}} & \left[T\begin{pmatrix}-1\\2\end{pmatrix}\right]_{\mathcal{B}} \\ | & | \end{pmatrix}$$

Ako vektore baze $\mathcal{B} = \left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}$ označimo sa u_1 i u_2

tj. sa $u_1 = \begin{pmatrix}2\\-1\end{pmatrix}$, $u_2 = \begin{pmatrix}-1\\2\end{pmatrix}$, da bi odredili $[T(u_1)]_{\mathcal{B}}$ i $[T(u_2)]_{\mathcal{B}}$

koristimo formule

$$[T(u_1)]_{\mathcal{B}} = [T]_{\varphi\mathcal{B}} \cdot [u_1]_{\varphi}$$

$$[T(u_2)]_{\mathcal{B}} = [T]_{\varphi\mathcal{B}} \cdot [u_2]_{\varphi}$$

Znamo $[u_1]_{\varphi} = \begin{pmatrix}2\\-1\end{pmatrix}$ i $[u_2]_{\varphi} = \begin{pmatrix}-1\\2\end{pmatrix}$. Odredimo još $[T]_{\varphi\mathcal{B}}$

$$[T]_{\varphi\mathcal{B}} = \begin{pmatrix} | & | \\ [T(e_1)]_{\mathcal{B}} & [T(e_2)]_{\mathcal{B}} \\ | & | \end{pmatrix}$$

Tražimo α i β t.d. $\alpha\begin{pmatrix}2\\-1\end{pmatrix} + \beta\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} \Rightarrow$

$\Rightarrow$ ZA VJEŽBU ... $\Rightarrow \alpha = \frac{\sqrt{2}}{2}, \beta = \frac{\sqrt{2}}{2} \Rightarrow [T(e_1)]_{\mathcal{B}} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$

Pa sad tražimo npr γ i δ t.d. $\gamma\begin{pmatrix}2\\-1\end{pmatrix} + \delta\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix} \Rightarrow$

$\Rightarrow$ ZA VJEŽBU ... $\Rightarrow \gamma = \frac{\sqrt{2}}{6}, \beta = \frac{-\sqrt{2}}{6} \Rightarrow$

$$\Rightarrow \quad [T(e_2)]_{\mathcal{B}} = \begin{pmatrix} \frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{6} \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad [T]_{\mathcal{SB}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \end{pmatrix}$$

Na kraju imamo

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{5\sqrt{2}}{6} \\ \frac{7\sqrt{2}}{6} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{6} \\ -\frac{5\sqrt{2}}{6} \end{pmatrix}$$

Prema tome

$$[T]_{\mathcal{B}} = \begin{pmatrix} \frac{5\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} \\ \frac{7\sqrt{2}}{6} & -\frac{5\sqrt{2}}{6} \end{pmatrix}$$

(#) U unitarnom prostoru $\mathbb{R}^4$, sa skalarnim proizvodom

$$\langle x, y\rangle = x_1 y_1 + 2x_2 y_2 + x_3 y_3 + 2 x_4 y_4$$

zadan je podprostor $\mathcal{V}$ razapet vektorima $v_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ i $v_2 = (1,0,1,1)^T$. Prikažite vektor $x = (4,2,2,4)^T$ u obliku $x = v + w$, gdje je $v \in \mathcal{V}$, $w \in \mathcal{V}^\perp$.

Rj.

Prisjetimo se

<u>Ortogonalni komplement</u>

<u>Za podskup $\mathcal{M}$ unitarnog prostora $\mathcal{V}$, ortogonalni komplement $\mathcal{M}^\perp$ od $\mathcal{M}$ je definisan sa</u>

$$\underline{\mathcal{M}^\perp = \{x \in \mathcal{V} \mid \langle m, x\rangle = 0 \text{ za } \forall m \in \mathcal{M}\}}.$$

U našem slučaju primjetimo da je

$$\mathcal{V} = \operatorname{span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\end{pmatrix}\right\}$$

Kako je $\dim \mathbb{R}^4 = 4$, $\dim \mathcal{V} = 2$ i $\mathbb{R}^4 = \mathcal{V} \oplus \mathcal{V}^\perp$ to je $\dim \mathcal{V}^\perp = 2$. Odredimo vektore $v_3 = (a,b,c,d)$ i $v_4 = (e,f,g,h)$ takve da

$$\begin{aligned}
\langle v_1, v_3\rangle = 0 &\Rightarrow a + c = 0\\
\langle v_2, v_3\rangle = 0 &\Rightarrow a + c + 2d = 0\\
\langle v_1, v_4\rangle = 0 &\Rightarrow e + g = 0\\
\langle v_2, v_4\rangle = 0 &\Rightarrow e + g + 2h = 0
\end{aligned}$$

$$\begin{aligned} a + c &= 0\\ a + c + 2d &= 0 \end{aligned}$$

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 2 & 0\end{array}\right] \overset{II_v - I_v}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & 0\end{array}\right] \Rightarrow \begin{aligned} a &= -c\\ d &= 0\end{aligned}$$

$$\left.\begin{matrix} rang(A) = rang(\bar{A}) = 2 \\ broj\ nepoznatih = 4 \end{matrix}\right\} \Rightarrow$$ dvije promjenjive uzimamo proizvoljno npr. $b=t$, $c=s$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s \\ t \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} t$$

Primjetimo da smo u procesu određivanja vektora v_3 u stvari odredili i vektor v_4. Time smo dobili:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Lagana provjera nam pokazuje da je $v_1 \perp v_3$, $v_1 \perp v_4$, $v_2 \perp v_3$, $v_2 \perp v_4$.

Da li je skup $\{v_1, v_2, v_3, v_4\}$ linearno nezavisan?

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \underset{III_v - I_v}{\overset{II_v \leftrightarrow IV_v}{\sim}} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$ skup $\{v_1, v_2, v_3, v_4\}$ jest linearno nezavisan.

Ostalo je još da razložimo vektor x preko vektora v_1, v_2, v_3, v_4. Odredimo α, β, γ i δ t.d.

$$\begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \left.\begin{matrix} \alpha + \beta - \gamma = 4 \\ \alpha + \beta + \gamma = 2 \end{matrix}\right\} \overset{+}{\Rightarrow} 2\alpha + 2\beta = 6 \\ \delta = 2, \ \beta = 4 \quad 2\alpha + 8 = 6 \\ \alpha = -1 \end{matrix}$$

$$\Rightarrow \alpha = -1, \ \beta = 4, \ \gamma = -1, \ \delta = 2$$

Možemo zaključiti da je

$$\underbrace{\begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 \\ 0 \\ 3 \\ 4 \end{pmatrix}}_{v} + \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}}_{w}$$ gdje su $v = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 4 \end{pmatrix} \in V$ i $w = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \in V^{\perp}$